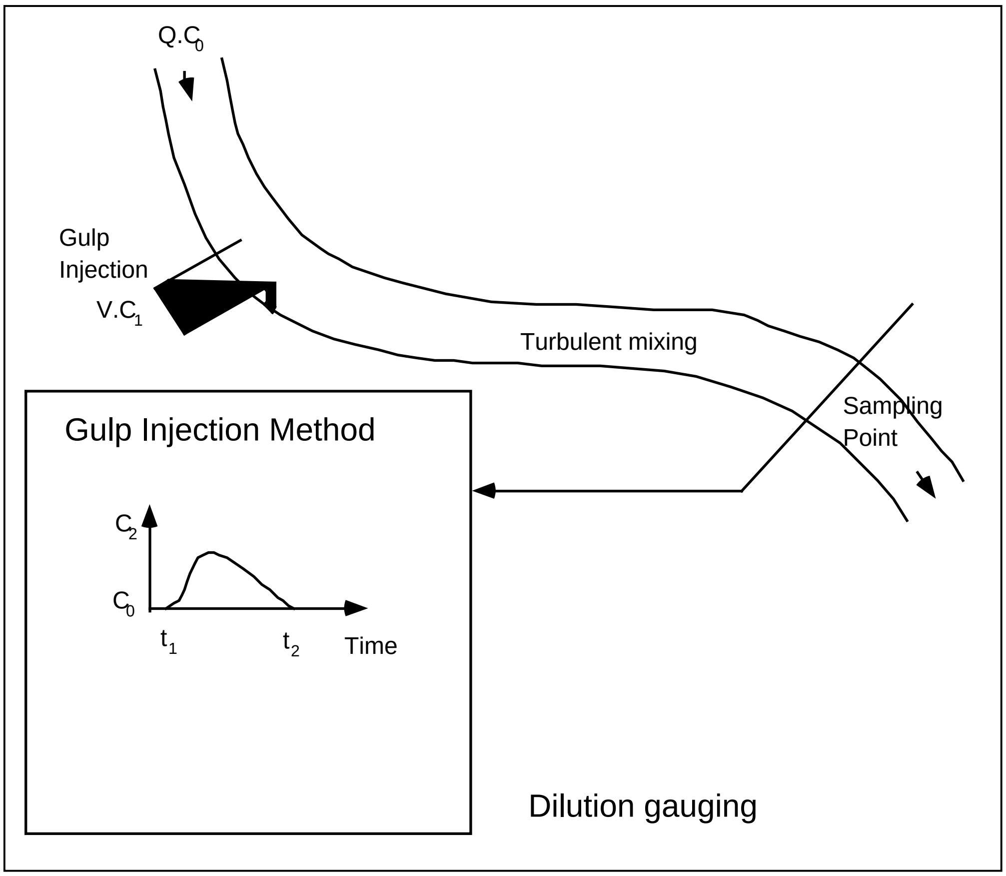
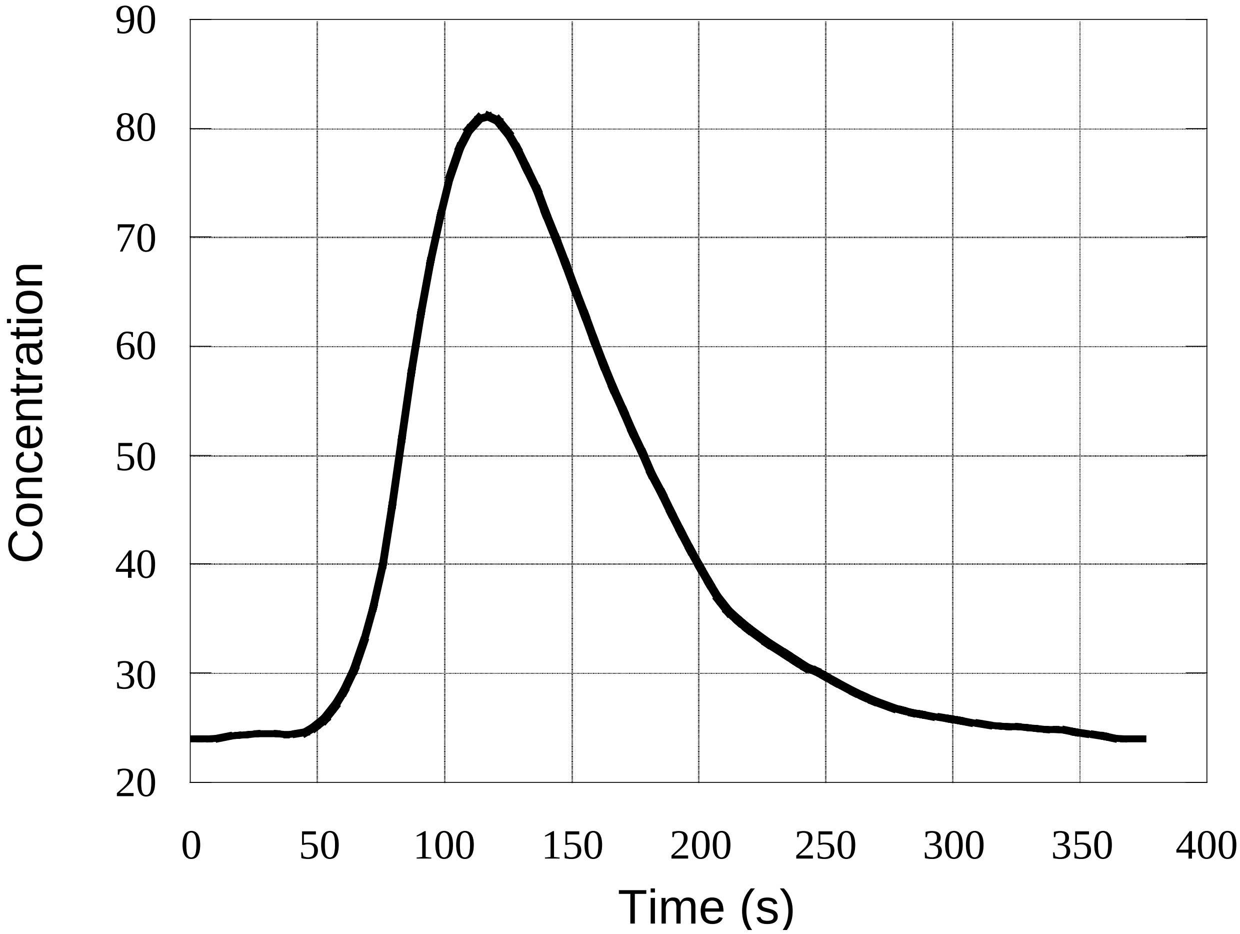
3 Area Under a Curve

3.1 Introduction

The measurement of the discharge rate, Q, of streams and rivers can often be difficult to measure. This often means the building of weirs and the measurement of water depths and velocities. This involves the building of infrastructure and is therefore limited to specific locations. An alternative method is to carry out a procedure called **dilution gauging**. Here a known volume of a tracer, such as common salt, is added to the flow.



In the figure C_0, C_1, and C_2 are chemical concentrations, where C_0 is the background concentration. In the **gulp injection method** a known volume V, of tracer concentration C_1 is instantaneously added to the stream and, at the sampling point the varying concentration C_2 is measured at regular time intervals. The figure below shows the data from a real experiment.



We can then write an equation relating the source tracer solution to the river flow by assuming mass conservation,

Mass of tracer in source = Mass of tracer in river

$$V*C_1 = Q*(C_2-C_0)*t$$

Test dimensions,

$$L^3 \times \frac{M}{L^3} = \frac{L^3}{T} \times \frac{M}{L^3} \times T$$

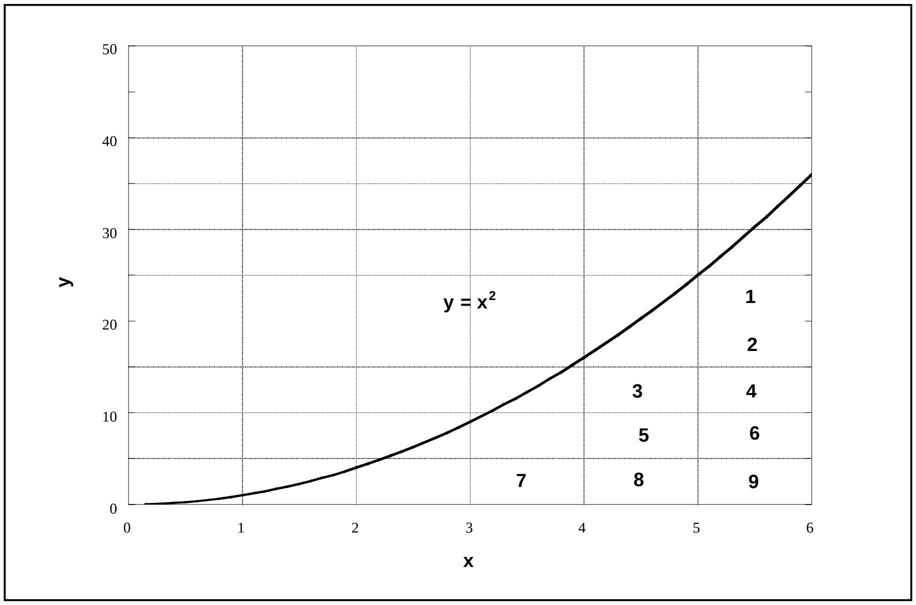
$$M = M$$

From the plot of t versus (C_2-C_0) you can see that (C_2-C_0) varies with t. If (C_2-C_0) had been a square wave then you can see that the value of (C_2-C_0)*t is the **area under the curve**, i.e. the area between the line and the t-axis, [SKETCH ON BOARD].

This also holds when (C_2-C_0) varies with t. If we can find the **area under the curve** then we can get a value for the flow rate of the river from this method.
You will be calculating flow rate of a river from dilution gauging during workshop 4.

3.2 Counting the Squares

One way of finding the area is by counting the squares within the confines of the curve and axis. This method requires you to:



1. decide on the size of square to use,
2. and what is the smallest fraction you can reliably count.

2.1 Size of Square

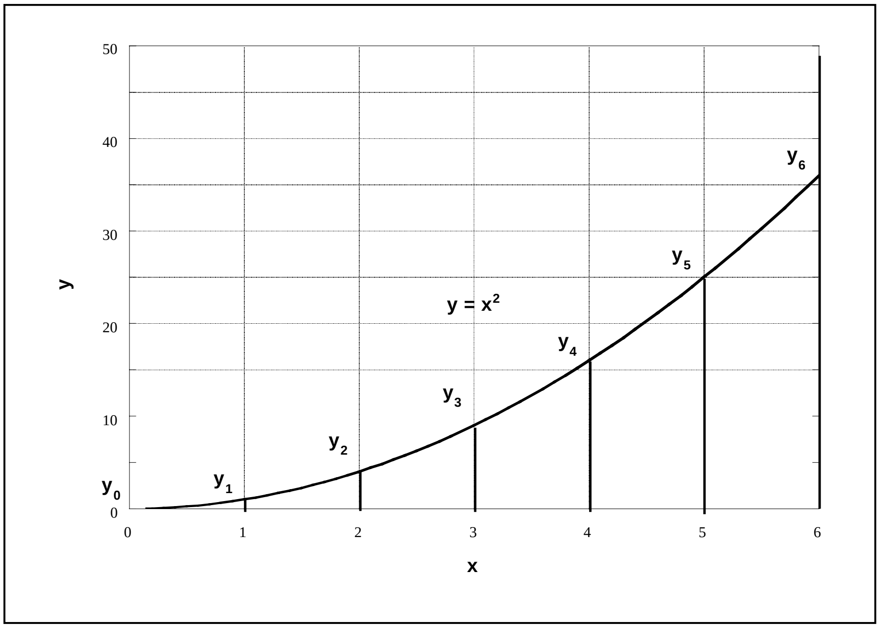
The size of the square chosen often depends of the size of square of the graph paper used. In general terms square size is a balance between an accurate estimate of area using small squares and a quicker counting time using large squares. In the figure above for the curve $y = x^2$, for example, the squares are large. If we counted only whole squares we get a count of 9. Each square has an area, $x*y$, of $1*5$ giving a counted area of 45 units. This will obviously be an underestimate of the true value.

2.2 Fraction of a Square

We can also count fractions of a square. Estimating 1/2 squares is usually quite reliable and we can 'guesstimate' the partial squares. If we do this for our concentration data we now get a total of about 14 squares giving and area of 70 units. This is considerably larger than the previous estimate based on whole squares because we chose large squares initially.

3.3 The Trapezium Rule

Instead of counting squares we can divide the area under the curve with a series of vertical strips. Where the vertical lines intersect the curve a chord is drawn creating a **trapezium**.



If the width of the strip and the height of the two vertical sides are known then the area can be found from the formula,

$$\text{area} = \frac{1}{2}(\text{sum of } \parallel \text{ sides}) \times \text{width}$$

If we are finding the area under a y = mx + c graph then we only need one trapezium and we get an exact value for the area. However, for curved lines the sum of the area of all the strips then gives an approximate value for the area under the curve. This value will be an over estimate for concave curves and an underestimate for convex curves. Now suppose that there are *n* strips, *all with the same width*, *d* say, and that the vertical edges of the strips (i.e. the ordinates) are labelled y_0, y_1, y_2, ... y_{n-1}, y_n.

The sum of the areas of all the strips can be written down as follows:

$$0.5(y_0 + y_1)(d) + 0.5(y_1 + y_2)(d) + ... +0.5(y_{n-2} + y_{n-1})(d) + 0.5(y_{n-1} + y_n)(d)$$

Therefore the area, A, under the curve is given approximately by

$$A = \frac{1}{2}(d)\left[y_0 + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n\right]$$

This formula is known as the **trapezium rule**. An easy way to remember the formula in terms of ordinates is,

half width of strip∗(first + last + twice all the others)

From the figure of $y = x^2$ we have a *d* value of 1 and seven values of y (y_0 to y_6). These values can be read from the graph or calculated from the equation as 0, 1, 4, 9, 16, 25 and 36 respectively. Putting these into the above equation gives,

$$A = \frac{1}{2}(1)\left[0 + 2 + 8 + 18 + 32 + 50 + 36\right]$$

Which gives A = 73. This is likely to be a better estimate of the area than our attempt at counting whole squares because it approximates all those parts of squares not counted. The value is very close to our 'guesstimated' partial square count. The trapezium rule value in this case will be an overestimate because of the concave nature of the curve.

3.4 Simpson's Rule

A formula which gives a better approximation than that obtained from the trapezium rule is known as *Simpson's rule*. The trapezium rule uses a straight line chord to join the vertical strips. This creates errors if the plotted line is highly curved. Simpson's rule joins three adjacent verticals with a parabola. This rule can only be used with an even number of strips, i.e. and odd number of ordinates. The formula for Simpson's rule is,

$$A = \frac{1}{3}d\left[\{y_0 + y_n\} + 4\{y_1 + y_3 + ...\} + 2\{y_2 + y_4 + ...\}\right]$$

Evaluating for our $y = x^2$ example gives,

$$A = \frac{1}{3} 1 \left[\{0 + 36\} + 4\{1 + 9 + 25\} + 2\{4 + 16\} \right]$$

Which gives A = 72. This is likely to be a better estimate of the area than the trapezium rule.

3.5 Getting Smaller

You can see that for all three of these methods that the smaller the square or value of d, the more accurate the area estimate becomes. The problem is that counting and calculating all those numbers is very time consuming, and a computer is really required to make a good job of this.

Example 3.1

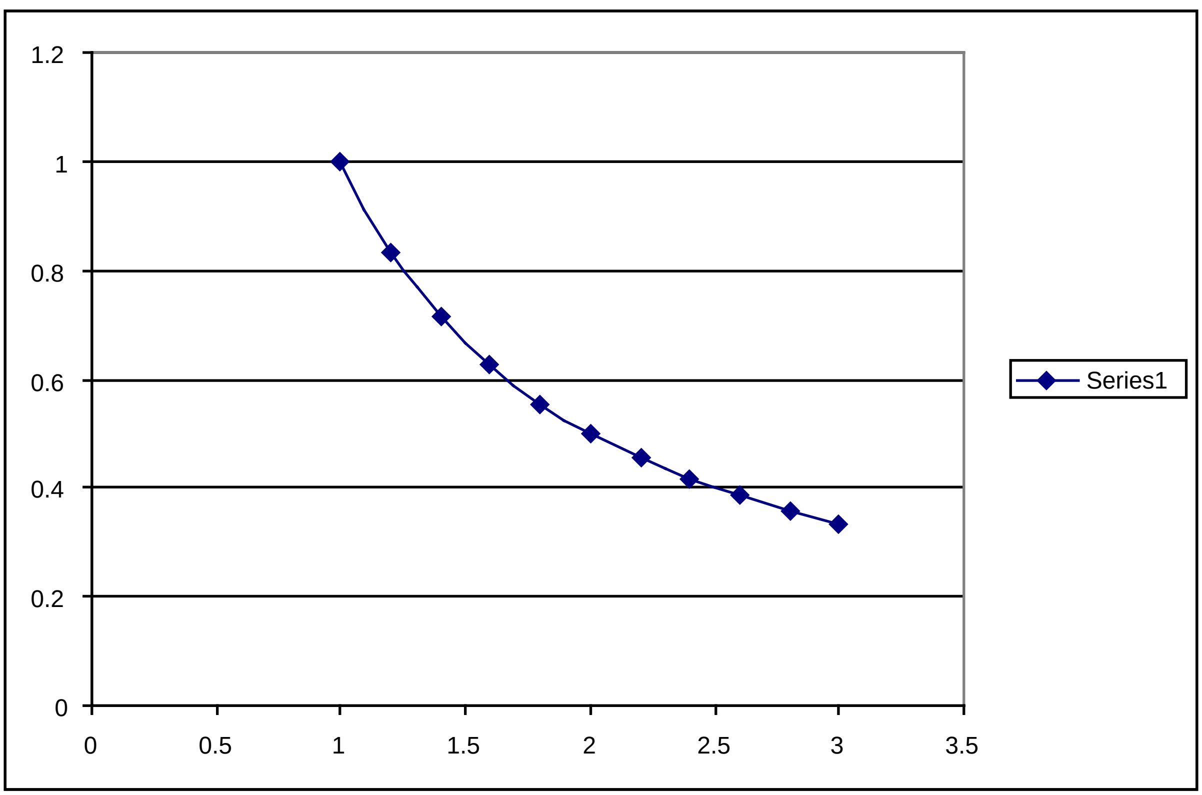
Estimate the area under the curve $y = \frac{1}{x}$ between $x = 1$ and 3, using

(a) the trapezium rule (b) Simpson's rule

with 11 ordinates, i.e. 10 strips

Answer

With 10 strips from 1 to 3, the strip width is 0.2. The curve is $y = \frac{1}{x}$ (see fig. Below)



and so the following points are calculated.

Suffix	x	Y=1/x
0	1.0	1=1/1
1	1.2	0.83333 = 1/1.2
2	1.4	0.71429 = 1/1.4
3	1.6	0.62500 = 1/1.6
4	1.8	0.55556 = 1/1.8
5	2.0	0.50000 = 1/2
6	2.2	0.45455 = 1/2.2
7	2.4	0.41667 = 1/2.4
8	2.6	0.38462 = 1/2.6
9	2.8	0.35714 = 1/2.8
10	3.0	0.33333 = 1/3

Using the trapezium rule

$$A = \frac{1}{2}(d)\left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_9 + y_{10}\right]$$

$$\therefore A = 0.5(0.2)\left[y_0 + y_{10} + 2(y_1 + y_2 + \ldots + y_9)\right]$$

$$\therefore A = 0.5(0.2)\left[1.0 + 0.33333 + 2(4.84116)\right] = 1.101565$$

Simpson's rule is written as

$$A = \frac{1}{3}d\left[\left\{y_0 + y_{10}\right\} + 4\left\{y_1 + y_3 + y_5 + y_7 + y_9\right\} + 2\left\{y_2 + y_4 + y_6 + y_8\right\}\right]$$

$$\therefore A = \frac{1}{3}(0.2)\left[1 + 0.33333 + 4(2.73214) + 2(2.10902)\right] = 1.098662$$

Using the definite integral it can be evaluated exactly to give

$$\int_1^3 \frac{1}{x}\,dx = \left[\ln x\right]_1^3 = \ln 3 - \ln 1 = 1.098612 - 0 = 1.098612$$

and so it is possible to compare the effectiveness of these two methods on this calculation.

Exact value	1.098612	Error	Error (%)
Trapezium rule	1.101565	0.002953	0.27%
Simpson's rule	1.098662	0.000050	0.0045%

Both methods used 10 strips each width 0.2. The greater accuracy of Simpson's rule means that for a specified degree of accuracy in the calculation fewer strips are required with Simpson's rule than with the Trapezium rule. In fact, using 4 strips, each of width 0.5, Simpson's rule gives 1.10000 as the area which is in error by 0.001388 or 0.13%. This error is still half the size of that due to the trapezium rule using 10 strips.